Lecture 03. Probability

Probability is a measure of likeliness on the occurrence of an event, ranging from 0 to 1 (100%). Here, of course, we need a clear definition of the event. For examples,

(1) Tomorrow, the event that 'it will rain at Taichung area' has a probability of 40%.

(2) 'A man of age older than 50 will suffer from hypertension problem' has probability of 27%. (Prevalence)

(3) A patient with heart disease and having transplantation has a probability of 58% for surviving over 5 years (after his/her transplant).

So, it's crucial to give a totally clear definition, which usually depends on an elucidation of an 'operation procedure', on an event.

☆ Frequentist definition of probability of an event:

1. If a well-defined event, A, can be repeated under virtually identical conditions (under which the event is defined), and it occurs m times in n replicates, the ratio m/n approaches a fixed limit as n goes to infinity, i.e.,

Pr (event A occurs) or simply Pr(A) or $P(A)=m/n$

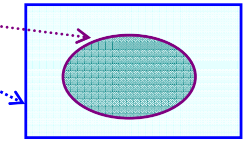
2. If the stated fixed limit is p, $p=\lim_{\{n\to\infty\}} m/n$, then in any finite n replicates, the expected (or mean) number in which event A will occur is np.

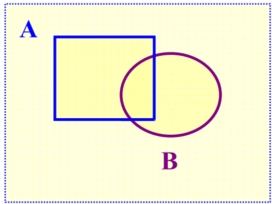
Notations, definitions, and Venn diagram

- **Complement**: $A^C=\Omega - A$

$\Longrightarrow$ $P(A^C)=1-P(A)$

$(P(\Omega)=1)$





- **Intersection**: $A\cap B$
- **Union**: $A\cup B$

$\Longrightarrow$ $P(A\cup B)=P(A)+P(B)-P(AB)$

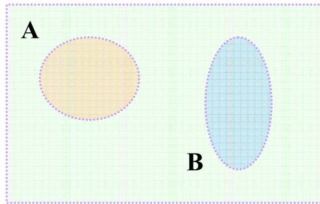
$A-B=\{w: w \text{ is in } A \text{ and } w \text{ is not in } B\}$

$A=(A-B)\cup(A\cap B)$

- **Mutually exclusive; disjoint :**

$A\cap B=\Phi$; $P(A\cap B)=0$;

$\Longrightarrow$ $P(A\cup B)=P(A)+P(B)$



- **Conditional Probability**

$P(A \cap B) = P(AB) = P(A)P(B|A) = P(B)P(A|B)$

$P(B|A)$: the probability of event B occurs, conditional on (or given) event A occurred.

Useful formula:

$$P(A|B)=P(AB)/P(B),$$

$$\text{or, } P(B|A)=P(AB)/P(A)$$

- **Independent events**

A and B are independent if

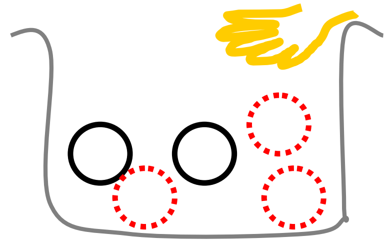
$P(AB) = P(A)P(B)$, or

$P(A|B) = P(A)$, or $P(B|A) = P(B)$

NOTE:
$P(AB) = P(A)P(B|A) = P(A)P(B)$.
If A and B are independent, that will imply
$P(B|A)=P(B)$

NOTE: independence vs. disjoint (mutually exclusive)
Independence doesn't imply exclusiveness. (See the example below.) On the other hand, if two events are disjoint, that means the occurrence of one event will certainly depend on that of the other.

Example 1:



To draw two balls without replacement and one by one.

Q: Pr(2 red balls)=?

A: Intuitively, the answer is

$${}_3C_2 \div {}_5C_2 = [3!/2!(3-2)!] \div [5!/2!(5-2)!] = 3/10$$

Now, because Pr(the first drawn is red)=3/5

[thus Pr(the first drawn is black)=2/5];

$\Rightarrow$ **Pr(2 red balls)**

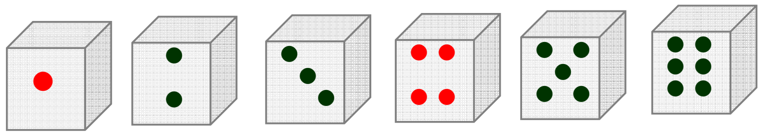
=Pr(the 1st drawn is red) ×

Pr(the 2nd drawn is red | the 1st drawn is red)

=3/5 × 2/4 = 3/10

Homework:
Pr(the 2nd drawn is red)=?
Pr(the 5th drawn is red)=?

Example 2: Tossing a fair dice, let X=number, Y=color, Z=Odd or Even.



Pr(Y=Red, Z=Odd)=Pr{X=1}=1/6;

Pr(Y=Red)=2/6; Pr(Z=Odd)=3/6, we have

Pr(Y=Red, Z=Odd)= Pr(Y=Red)×Pr(Z=Odd)

We say: the event "red" is independent of "odd".

Comment: Although we are able to define 'independence' between events through the above expression, the meaning and interpretation of independent events sometimes have intuitive resorts and feasibility of daily-life experience.

Homework:

Please give at least two actual (and non-trivial) examples to explain 'independence'

Bayes' Theorem

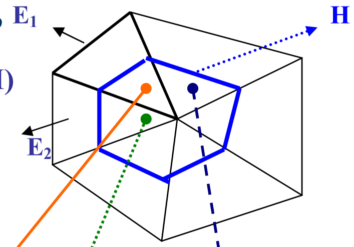
- **Partition** of the universal event (set) Ω :

$$\Omega=E_1\cup E_2\cup E_3\cup\ldots\cup E_k$$

$E_1,E_2,E_3,\ldots,E_k$ Are mutually exclusive

That is, $E_i\cap E_j=\Phi$, for all $i\neq j$

- H is a subset of Ω,



$P(E_1|H)=P(E_1H)/P(H)$

$= P(HE_1)/P(H)$

$= P(H|E_1)P(E_1)/P(H)$

$$P(H)=P(H\cap\Omega)$$
$$=P(H\cap(E_1\cup E_2\cup E_3\cup\ldots\cup E_k))$$
$$=P\{(H\cap E_1)\cup(H\cap E_2)\cup\ldots\cup(H\cap E_k)\}$$
$$=P(H\cap E_1)+P(H\cap E_2)+\ldots+P(H\cap E_k),\ (\text{why??})$$
$$=P(H|E_1)P(E_1)+P(H|E_2)P(E_2)+\ldots+P(H|E_k)P(E_k)$$
$$=\Sigma_i P(H|E_i)P(E_i)$$

$=P(H|E_1)P(E_1) / \Sigma_i P(H|E_i)P(E_i)$

APPLICATION of BAYES' THEOREM: Diagnostic test

- **Sensitivity (sens.): $P(T+|D+)$**
- **Specificity (spec.): $P(T-|D-)$**

$\Big\}$ usually estimated from a group of people with known disease-status

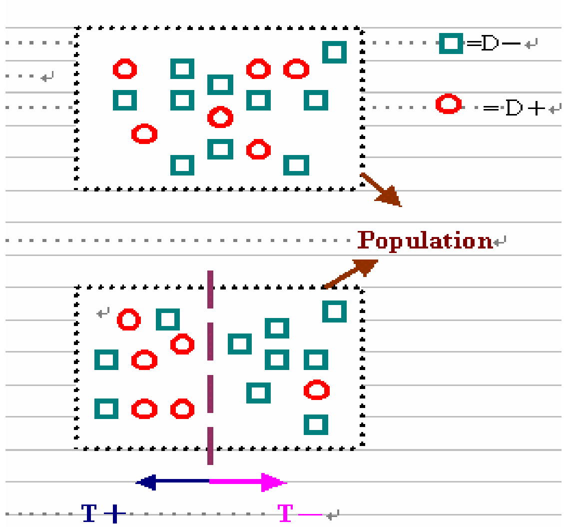
- **False positive (FP)=$P(T+|D-)=1-P(T-|D-)=1-$spec.**
- **False negative (FN) =$P(T-|D+)=1-P(T+|D+)=1-$sens.**
- **Prevalence : $P(D+)$, proportion of diseased people in general population (prior probability); often obtained from vital statistics or a large sample survey (maybe with random sampling), or some other statistical sampling methods and estimation.**
- $P(D+|T+)$, **posterior** probability

  $= P(T+|D+)P(D+)/\ P(T+|D+)P(D+)+ P(T+|D-)P(D-)$

  = Sens*Prev/ Sens*Prev+FP*(1-Prev)


Illustration

下圖中每一個圈圈（ 表有病者，D+）或者方塊（ 表沒病者，D－）均代表一萬人(10,000) ；外圍之虛線方框代表母群體(population)。據此，該族群(population)共有 160,000 人；有病者共 60,000 人，沒病者共 100,000 人。



⇨ 該疾病在族群中之盛行率(prevalence)=6/16

Sensitivity=5/6; Specificity=7/10

$Pr(D+ | T+)=(5/6) * (6/16) \div [(5/6) * (6/16)+ (1-7/10)(1-6/16)]$

$= 5 / 16 \div [5 / 16 + 3 / 16]$

$= 5 / 8$

Homework: Pr (D- | T-)=…., please check it, =7 / 8

Example 3: (HIV infection)

Sensitivity: Pr(HIV test + | infected)=0.98

Specificity: Pr(HIV test - | not infected)=0.9999

If Pr(infected)=5000/2×10^7 =2.5/10^4 , prevalence

⇨ **Pr(infected | HIV test +)**

= Pr(HIV test + | infected) × Pr(infected) ÷

[Pr(HIV test + | infected) × Pr(infected) +

Pr(HIV test + | not infected) × Pr(not infected)]

= 0.98 × 0.00025 ÷

[0.98 × 0.00025 + (1-0.9999)×(1-0.00025)]

= **0.71**

Homework and exercises:

Problems in your textbook [pp. 80~87]
1, 3, 7, 8, 11, 14, 17, 26, 28, 34, 38, 39, 43